The University of Georgia
Department of Mathematics and Science Education
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EMAT 6680 - Assignment 3

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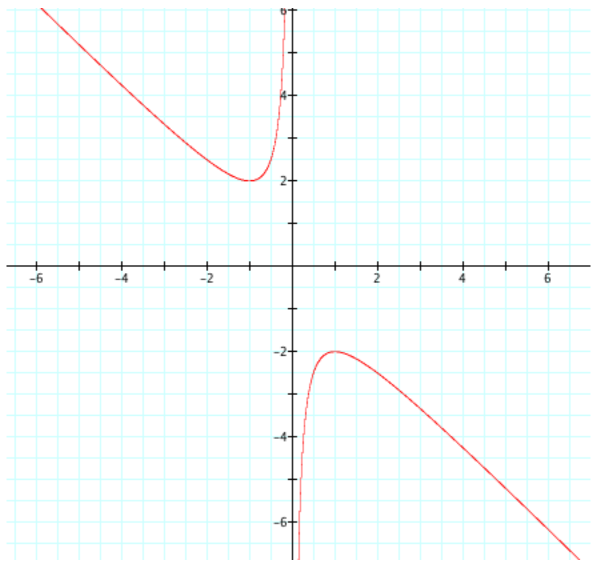
Question: Explore solutions to quadratic equations using graphing calculator.

Let's begin by considering the quadratic equation $x^2 + bx + 1 = 0$. The solutions to such an equation in the xb plane are graphed below.



By observing the graph, we conjecture that there are no real solutions to the equation whenever $|b| < 2$, one real solution when $|b| = 2$, and two real solutions when $|b| > 2$. Indeed, by using the quadratic formula we can actually prove this.

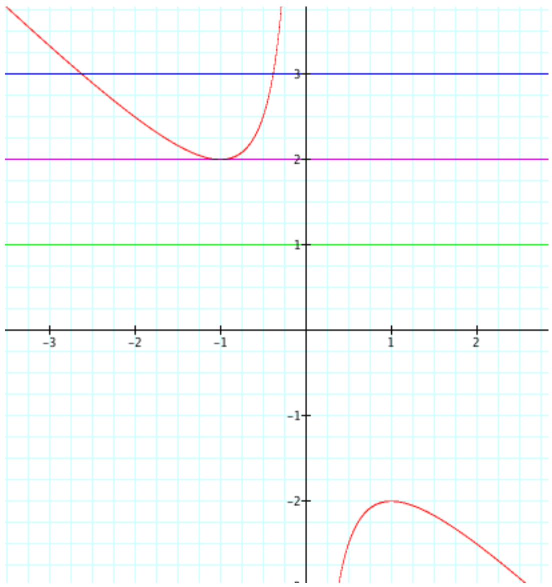
Lemma 1. *The quadratic equation $x^2 + bx + 1 = 0$ has no real solutions to the equation whenever $|b| < 2$, one real solution when $|b| = 2$, and two real solutions when $|b| > 2$.*

Proof. The solutions to the equation $x^2 + bx + 1 = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4}}{2}.$$

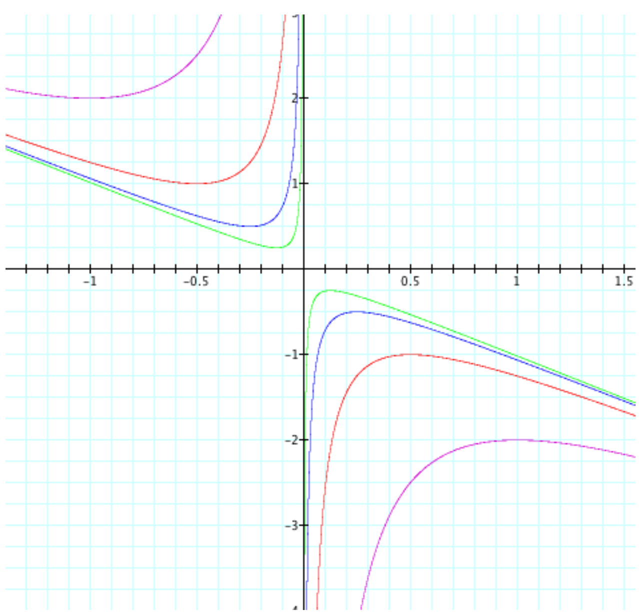
First, if $|b| = 2$, then $b^2 = 4$, i.e., $b^2 - 4 = 0$, so the equation has a repeated solution $x = \frac{-b}{2}$. If $|b| < 2$, then $b^2 < 4$ and there is a negative number underneath the radical, so no real solutions exist. Finally, if $|b| > 2$, then $b^2 > 4$, so there are two real solutions. □

Now, we said that looking at the graph indicates the there are either 0, 1, or 2 solutions depending on the value for b, but how does this follow from the graph? Let's begin by graphing various lines $b = K$ for some fixed constants $K = 1$ (green), $K = 2$ (purple), $K = 3$ (blue).



Now, the green line does not intersect the graph in any points, which indicates that there are no solutions to the equation for that value of K. Moreover, we can see that no horizontal line between $b = -2$ and $b = 2$ will intersect the curve, so there are no real solutions. The purple line intersects the graph in exactly one point, which indicates the repeated solution $x = \dfrac{-b}{2}$. Of course, the blue line intersects the curve twice, which matches the two real solutions.

Next, we would like to understand what happens as the c-term changes. Can we make it be the case that there is always at least 1 real solution? By the above quadratic equation, we see that we want $b^2 - 4c \geq 0$ for all $b \in \mathbb{R}$, so this tells us that we need $b^2 \geq 4c$ for all $b \in \mathbb{R}$. Of course, this will be satisfied as long as $c \leq 0$. As c tends to zero, we see that there are more acceptable values for b.



Then we see exactly what we should expect for $c \leq 0$. Exactly one real solution for $b \in \mathbb{R}$ when $c = 0$ and two real solutions for all $b \in \mathbb{R}$ when $c < 0$.